ON k-CAPS IN $\mathbf{PG}(n, q)$, WITH q EVEN AND $n \geq 4$

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ABSTRACT. Let $m_2(n, q)$, $n \geq 3$, be the maximum size of k for which there exists a complete k-cap in $\mathbf{PG}(n, q)$. In this paper the known bounds for $m_2(n, q)$, $n \geq 4$, q even and $q \geq 2048$, will be considerably improved.

Keywords: projective space, finite field, k-cap.

1. INTRODUCTION

A *k-arc* of $\mathbf{PG}(2, q)$ is a set of k points, no three of which are collinear; a *k-cap* of $\mathbf{PG}(n, q)$, $n \geq 3$, is a set of k points, no three of which are collinear. A k-arc or k-cap is *complete* if it is not contained in a $(k+1)$-arc or $(k+1)$-cap. The largest value of k for which a k-arc of $\mathbf{PG}(2, q)$, or a k-cap of $\mathbf{PG}(n, q)$ with $n \geq 3$, exists is denoted by $m_2(n, q)$. The size of the second largest complete k-arc of $\mathbf{PG}(2, q)$ or k-cap of $\mathbf{PG}(n, q)$, $n \geq 3$, is denoted by $m_2'(n, q)$.

For any k-arc K in $\mathbf{PG}(2, q)$ or k-cap K in $\mathbf{PG}(n, q)$, $n \geq 3$, a *tangent* of K is a line which has exactly one point in common with K. Let t be the number of tangents of K through a point P of K and let $\sigma_1(Q)$ be the number of tangents of K through a point $Q \notin K$. Then for a k-arc K $t + k = q + 2$ and for a k-cap K $t + k = q^{n-1} + q^{n-2} + \cdots + q + 2$.

Theorem 1.1 ([6]). *If K is a complete k-arc in $\mathbf{PG}(2, q)$, q even, or a complete k-cap in $\mathbf{PG}(n, q)$, $n \geq 3$ and q even, then $\sigma_1(Q) \leq t$ for each point Q not in K.*

Theorem 1.2. (i) $m_2(2, q) = q + 2$, *q even* [5];
(ii) $m_2(3, q) = q^2 + 1$, *q even, $q > 2$* [4, 1, 8];
(iii) $m_2(n, 2) = 2^n$ [1];
(iv) $m_2(4, 4) = 41$ [3];
(v) $m_2'(n, 2) = 2^{n-1} + 2^{n-3}$ [2];
(vi) $m_2'(3, 4) = 14$ [6].

Theorem 1.3 ([9, 11, 5]). *Let K be a k-arc of $\mathbf{PG}(2, q)$, q even and $q > 2$, with $q - \sqrt{q} + 1 < k \leq q + 1$. Then K can be uniquely extended to a $(q+2)$-arc of $\mathbf{PG}(2, q)$.*

The following result is the Main Theorem of [12].

Theorem 1.4 ([12]).

(1) $$m_2'(3, q) < q^2 - (\sqrt{5} - 1)q + 5, q \text{ even}, q \geq 8.$$

As a corollary new bounds for $m_2(n,q)$, q even, $q \geq 8$ and $n \geq 4$, are obtained.

Theorem 1.5 ([12]). (i) *$m_2(4,8) \geq 479$;*

(ii) *for q even, $q > 8$,*
$m_2(4,q) < q^3 - q^2 + 2\sqrt{5}q - 8$;

(iii) $m_2(n,8) \leq 478.8^{n-4} - 2(8^{n-5} + \cdots + 8 + 1) + 1, n \geq 5$;

(iv) *for q even, $q > 8, n \geq 5$,*
$m_2(n,q) < q^{n-1} - q^{n-2} + 2\sqrt{5}q^{n-3} - 9q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1.$

Combining the main theorem of [10] with Theorem 1.4, there is an immediate improvement of the upper bound for $m_2'(3,q), q \geq 2048$. This important remark is due to T. Szőnyi.

Theorem 1.6 ([12]).

$$m_2'(3,q) < q^2 - 2q + 3\sqrt{q} + 2, q \text{ even }, q \geq 2048. \tag{2}$$

Relying on Theorem 1.6, in the underlying paper new bounds for $m_2(n,q)$, q even, $q \geq 2048, n \geq 4$, will be obtained.

Theorem 1.7. *For q even, $q \geq 2048$,*

(i) $m_2(4,q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$,

(ii) $m_2(n,q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4}$
$- 2(q^{n-5} + \cdots + q + 1) + 1, n \geq 5.$

2. New bound for $m_2(4,q)$

Theorem 2.1. *For q even, $q \geq 2048$,*

$$m_2(4,q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6. \tag{3}$$

Proof. Assume, by way of contradiction, that K is a complete k-cap of $\mathbf{PG}(4,q)$ with

$$k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6. \tag{4}$$

Then

$$t \leq q^3 + q^2 + q + 2 - q^3 + 2q^2 - 3q\sqrt{q} - 8q + 9\sqrt{q} + 6, \tag{5}$$

so

$$t \leq 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8. \tag{6}$$

We obtain a contradiction in several stages.

(I) K **contains no plane q-arc**

Assume, by way of contradiction, that π is a plane with $|\pi \cap K| = q$; let $\pi \cap K = Q$.

(a) **Suppose that $\delta_1, \delta_2, ..., \delta_5$ are distinct hyperplanes containing π, such that**

$$|\delta_i \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2, i = 1,2,...,5 \tag{7}$$

By Theorem 1.6 each $\delta_i \cap K$ can be extended to an ovoid O_i of $\delta_i, i = 1,2,...,5$. Hence $O_i \cap \pi$ is a $(q+1)$-arc $Q \cup \{N_i\}, i = 1,2,...,5$. Since Q is contained in two

$(q+1)$-arcs at least three of the points N_i coincide, say $N_1 = N_2 = N_3$. The joins of N_1 to the points of $\delta_i \cap K$, with $i = 1,2,3$, are tangents of K.
Hence

$$\sigma_1(N_!) \geq 3(q^2 - 3q + 3\sqrt{q} + 2) + q, \tag{8}$$

so

$$\sigma_1(N_1) \geq 3q^2 - 8q + 9\sqrt{q} + 6. \tag{9}$$

As K is complete, $\sigma_1(N_1) \leq t$. So

$$3q^2 - 8q + 9\sqrt{q} + 6 \leq 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8, \tag{10}$$

that is

$$3q\sqrt{q} - q - 2 \leq 0, \tag{11}$$

clearly a contradiction.

(b) **Assume that there are at most 4 hyperplanes δ of $\mathbf{PG}(4,q)$ containing π with** $|\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2$

Then counting points of K in hyperplanes containing π gives

$$k < (q-3)(q^2 - 3q + 3\sqrt{q} + 2) + 4(q^2 - q) + q, \tag{12}$$

that is,

$$k < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6. \tag{13}$$

(Remark that any hyperplane containing π, has at most q^2 points in common with K.)

But $k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$, clearly a contradiction.

(II) **There exists no hyperplane δ of $\mathbf{PG}(4,q)$ such that**

$$q^2 + 1 > |\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2 \tag{14}$$

Suppose, by way of contradiction, that such a δ exists. Let $\delta \cap K = K'$. Then K' can be extended to an ovoid O of δ. Let $N \in O \setminus K'$ and let $N' \in K'$. Consider the $q+1$ planes of δ containing the line NN'. Each of these planes meets O in a $(q+1)$-arc, so by I each such plane meets K' in at most a $(q-1)$-arc.

Assume, by way of contradiction, that none of these intersections is a $(q-1)$-arc. Counting the points of K' on these $q+1$ planes gives

$$|K'| \leq (q+1)(q-3) + 1, \tag{15}$$

so

$$|K'| \leq q^2 - 2q - 2. \tag{16}$$

As $|K'| \geq q^2 - 2q + 3\sqrt{q} + 2$, there arises $3\sqrt{q} + 4 \leq 0$, a contradiction.

So we may assume that $|\pi \cap K'| = q - 1, \pi \subset \delta, NN' \subset \pi$. Consider all hyperplanes of $\mathbf{PG}(4,q)$ containing the plane π. Let θ be the number of such hyperplanes π' for which

$$|\pi' \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2. \tag{17}$$

By assumption $\theta \geq 1$.

First assume $\theta \geq 4$, hence there are at least 4 hyperplanes $\pi'_1, \pi'_2, \pi'_3, \pi'_4$ containing π such that

$$|\pi'_i \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2. \tag{18}$$

Consequently $\pi'_i \cap K$ can be extended to an ovoid O_i of π'_i, with $i = 1, 2, 3, 4$. It follows that $O_i \cap \pi$ is a $(q+1)$-arc $(\pi \cap K) \cup \{N'_i, N''_i\}, i = 1, 2, 3, 4$. The $(q-1)$-arc $\pi \cap K$ is extendable to a unique $(q+2)$-arc R of π, and each $(q+1)$-arc of π containing $\pi \cap K$ belongs to R [5] . So $\pi \cap K$ is contained in exactly 3 $(q+1)$-arcs of π. It follows that there is at least one point N which belongs to 3 of the 4 pairs $\{N'_i, N''_i\}$. So the number of tangents $\sigma_1(N)$ of K containing N is at least

$$3(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + q - 1 = 3q^2 - 8q + 9\sqrt{q} + 8. \tag{19}$$

As $\sigma_1(N) \leq t$, there arises

$$3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 8 \geq t \geq \sigma_1(N) \geq 3q^2 - 8q + 9\sqrt{q} + 8, \tag{20}$$

so

$$3q\sqrt{q} - q \leq 0, \tag{21}$$

a contradiction.

Finally, assume $\theta \leq 3$. Counting the points of K in the $q+1$ hyperplanes containing π, we obtain

$$k < (q-2)(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + 3(q^2 - q + 1) + q - 1, \tag{22}$$

so

$$k < q^3 - 2q^2 + 3q\sqrt{q} + 7q - 6\sqrt{q} - 4. \tag{23}$$

As

$$k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6, \tag{24}$$

there arises

$$q - 3\sqrt{q} - 2 < 0, \tag{25}$$

a final contradiction.

(III) **For a point N not on K, there do not exist planes π_1 and π_2 such that $\pi_1 \cap \pi_2 = \{N\}$ and such that $\pi_i \cap K$ is a $(q+1)$-arc with nucleus $N, i = 1, 2$**

Suppose, by way of contradiction, that such planes π_1, π_2 exist. Let δ be a hyperplane containing π_1. Then $\delta \cap K$ contains the $q+1$ tangents of $\pi_1 \cap K$ through N and one tangent of $\pi_2 \cap K$ through N. So $\delta \cap K$ has at least $q+2$ tangents through N. Hence $|\delta \cap K| < q^2 + 1$.

Suppose that

$$|\delta \cap K| < q^2 - 2q + 3\sqrt{q} + 2 \tag{26}$$

for any such hyperplane δ. Counting points of K in hyperplanes containing π_1 gives

$$k < (q+1)(q^2 - 3q + 3\sqrt{q} + 1) + q + 1, \tag{27}$$

so

$$k < q^3 - 2q^2 + 3q\sqrt{q} - q + 3\sqrt{q} + 2. \tag{28}$$

As $k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$, there arises a contradiction.

Consequently there exists a hyperplane δ through π_1 for which

$$q^2+1 > |\delta \cap K| \geq q^2 - 2q + 3\sqrt{q} + 2, \tag{29}$$

contradicting II.

(IV) **The tangents of K through any point N not on K lie in a hyperplane**

Let δ be a hyperplane not containing N and let $\mathcal{V}$ be the set of the intersections of δ with all tangents of K through N. We will show that each point of $\mathcal{V}$ is on at least two lines contained in $\mathcal{V}$.

Let $R \in \mathcal{V}$ and let $r = RN$. Assume, by way of contradiction, that for at most one plane π containing r we have $|\pi \cap K| \geq q-1$. So

$$k \leq (q^2+q)(q-3) + (q+1), \tag{30}$$

that is,

$$k \leq q^3 - 2q^2 - 2q + 1. \tag{31}$$

As $k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$, there arises $3q\sqrt{q} + 10q - 9\sqrt{q} - 7 \leq 0$, a contradiction.

Hence we may assume that for distinct planes π, π' containing r we have

$$|\pi \cap K|, |\pi' \cap K| \in \{q-1, q+1\}. \tag{32}$$

(By I no plane intersects K in a q-arc.)

We distinguish two cases.

(a) **At least one of the planes π, π' intersects K in a $(q-1)$-arc**

Say $|\pi \cap K| = q-1$. Assume, by way of contradiction, that for no hyperplane δ' containing π we have $|\delta' \cap K| = q^2+1$. Counting points of K in hyperplanes containing π gives by II

$$k < (q+1)(q^2 - 2q + 3\sqrt{q} + 2 - q + 1) + q - 1, \tag{33}$$

that is,

$$k < q^3 - 2q^2 + 3q\sqrt{q} + q + 3\sqrt{q} + 2. \tag{34}$$

As $k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$, there arises $7q - 12\sqrt{q} - 8 < 0$, a contradiction.

So for at least one hyperplane δ' containing π we have $|\delta' \cap K| = q^2+1$. But then $|\pi \cap K| = q+1$, again a contradiction.

(b) $|\pi \cap K| = |\pi' \cap K| = q+1$

If N is the nucleus of both $\pi \cap K$ and $\pi' \cap K$, then there are two lines of $\mathcal{V}$ through R, namely $\pi \cap \delta$ and $\pi' \cap \delta$.

Therefore suppose that N is not the nucleus of $\pi \cap K$. If for at most one hyperplane δ' containing π we have $|\delta' \cap K| = q^2+1$, then counting points of K in hyperplanes containing π gives

$$k < q^2 + 1 + q(q^2 - 3q + 3\sqrt{q} + 1), \tag{35}$$

so

$$k < q^3 - 2q^2 + 3q\sqrt{q} + q + 1. \tag{36}$$

As $k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6$, there arises $7q - 9\sqrt{q} - 7 < 0$, a contradiction.

Consequently there are at least two hyperplanes δ_1 and δ_2 containing π for which $\delta_i \cap K = O_i$ is an ovoid, $i = 1, 2$. then there is a plane π_i of δ_i containing N such that N is the nucleus of the $(q+1)$-arc $\pi_i \cap O_i = K_i, i = 1, 2$. As N is not the nucleus of $\pi \cap K$, we have $\pi \neq \pi_1 \neq \pi_2 \neq \pi$. The tangents of K_i (which contain N) meet δ in the points of a line l_i containing R, with $i = 1, 2$, and $l_1 \neq l_2$..

Consequently each point of $\mathcal{V}$ is on at least two lines contained in $\mathcal{V}$.

If there existed two skew lines in $\mathcal{V}$, there would be two planes π_1' and π_2' on N, with $\pi_1' \cap \pi_2' = \{N\}$ and N the nucleus of the $(q+1)$-arcs $\pi_1' \cap K$ and $\pi_2' \cap K$. This is in contradiction with III. It follows that the lines of $\mathcal{V}$ all have a common point or all lie in a common plane. As each point of $\mathcal{V}$ is on at least two lines of $\mathcal{V}$, all lines of $\mathcal{V}$ lie in a plane. Hence $\mathcal{V}$ is subset of a plane, and so all tangents of K containing N lie in a hyperplane.

(V) **The final contradiction**

The final contradiction will be obtained by counting all tangents of K.

Consider the function

(37) $$G(x) = x(q^3 + q^2 + q + 2 - x).$$

It attains its maximum value for

(38) $$x = \frac{1}{2}(q^3 + q^2 + q + 2).$$

We have

(39) $$q^3 > k \geq q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6 > \frac{1}{2}(q^3 + q^2 + q + 2),$$

so

(40) $$kt = k(q^3 + q^2 + q + 2 - k) = G(k) > G(q^3) = q^3(q^2 + q + 2).$$

All tangents containing a point N not on K lie in a hyperplane, which contains at most $q^2 + 1$ points of K. An ovoid of a hyperplane containing N has exactly $q + 1$ tangents containing N. Hence N is contained in at most q^2 tangents of K.

Counting the pairs (N, l), with $N \notin K$, l a tangent of K containing N, there arises

(41) $$(q^4 + q^3 + q^2 + q + 1 - k)q^2 \geq ktq,$$

so

(42) $$(q^4 + q^3 + q^2 + q + 1 - q^3 + 2q^2 - 3q\sqrt{q} - 8q + 9\sqrt{q} + 6)q \geq kt,$$

so

(43) $$(q^4 + 3q^2 - 3q\sqrt{q} - 7q + 9\sqrt{q} + 7)q \geq kt > q^3(q^2 + q + 2),$$

that is,

(44) $$q^3 - q^2 + 3q\sqrt{q} + 7q - 9\sqrt{q} - 7 < 0,$$

a contradiction. ∎

3. NEW BOUND FOR $m_2(n,q), n \geq 5$

Theorem 3.1. *For q even,* $q \geq 2048, n \geq 5$

(45)
$$m_2(n,q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1.$$

Proof By 6.14(ii) of [7] for $n \geq 5$ and $q > 2$, we have

$$m_2(n,q) \leq q^{n-4} m_2(4,q) - q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1. \tag{46}$$

From Theorem 2.1 the result follows. ■

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